Ruelle's Operator and Conformal Measures with Applications in Fractal Geometry and Number Theory

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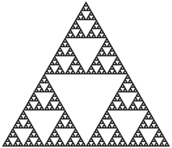


Uniwersity of Warsaw and Polish Mathematical Society

A Concept of Dynamical Systems

X–a set. $T : X \to X$–a map. Looking at the n-folded iterates

$$T^n := \underbrace{T \circ \ldots \circ T}_{n \text{ times}} : X \longrightarrow X$$

Fixed points:

$$T(x) = x$$

Periodic points

$$T^n(x) = x, \;\; n \in \mathbb{N}.$$

$$\limsup_{n\to\infty} \frac{1}{n} \log \#\mathrm{Per}_n(T), \;\; \liminf_{n\to\infty} \frac{1}{n} \log \#\mathrm{Per}_n(T), \;\; \lim_{n\to\infty} \frac{1}{n} \log \#\mathrm{Per}_n(T).$$

$A \subset X$:

$$\limsup_{n\to\infty} \frac{1}{n} \#\{0 \le k \le n-1 : T^k(x) \in A\} = \limsup_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \mathbb{1}_A \circ T^k(x).$$

Likewise, liminf and lim if it exists. Need of tools to calculate these quantities, especially the last one.

Invariant Measures

$(X, \mathfrak{F}, \mu)$–a probability space. $T : X \to X$–measurable. First attempt:

$$\mu(T(A)) = \mu(A)$$

Problems:

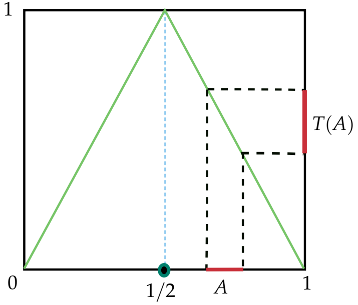
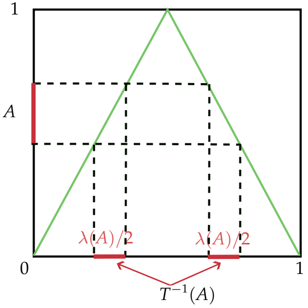
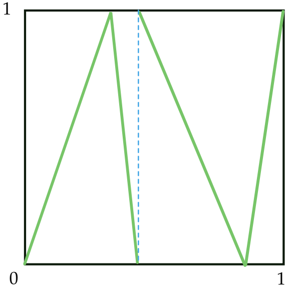


Figure: Tent map. We have that $\lambda(T(A)) = 2\lambda(A)$ where λ is the Lebesgue measure. No invariance.

Invariant Measures

$T(A)$ may not be measurable.





$$\lambda(T^{-1}(A)) = \lambda(A)$$

This appears to be the right definition.
In general, the measure μ is said to be T–invariant if and only if

$$\mu \circ T^{-1}(A) = \mu(A) \quad \text{for all sets } A \in \mathfrak{F}$$

Shorter: $\mu \circ T^{-1} = \mu$ (push-forward measure is the same)

Invariant Measures: Poincaré and Birkhoff

Theorem (Poincaré's Recurrence Theorem)

If $(X, \mathfrak{F}, \mu)$ *is a probability space,* $T : X \to X$ *is measurable, and* μ *is* T*–invariant, then for every measurable set* A

$$\mu\Big(\big\{x \in A : T^n(x) \in A \text{ for infinitely many } n\text{'s}\big\}\Big) = \mu(A).$$

Almost every point in A visits A infinitely often.

Definition (Ergodicity)

If there are no non–trivial backward invariant sets:

$$T^{-1}(A) = A \Longrightarrow \mu(A) = 0 \text{ or } 1,$$

then the map T (or the measure μ) is called ergodic.

Invariant Measures: Poincaré and Birkhoff

Theorem (Birkhoff's Ergodic Theorem)

Let $(X, \mathfrak{F}, \mu)$ be a probability space, $T : X \to X$ measurable, and μ T–invariant. If $f : X \to \mathbb{R}$ is integrable, then the limit

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} f \circ T^k(x)$$

exists for μ–a.e. $x \in X$. If T is ergodic, then

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} f \circ T^k(x) = \int_X f \, d\mu.$$

for μ–a.e. $x \in X$. The time average is equal to the space average. In particular, if $A \subset X$ is measurable, then

$$\lim_{n\to\infty} \frac{1}{n} \#\{0 \le k \le n-1 : T^k(x) \in A\} = \mu(A)$$

for μ–a.e. $x \in X$.

Quasi–Invariant Measures

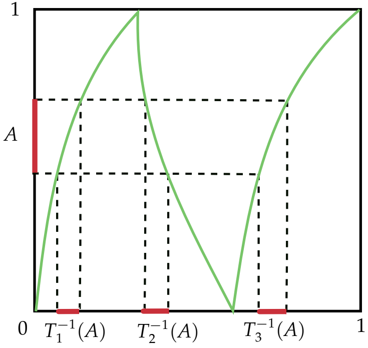


Figure: Here $\lambda(T^{-1}(A)) \neq \lambda(A)$.

$$\lambda(T_i^{-1}(A)) = \int_A |(T_i^{-1})'| \, d\lambda \implies \lambda(T^{-1}(A)) = \sum_{i=1}^{3} \int_A |(T_i^{-1})'| \, d\lambda$$

Quasi–Invariant Measures

μ absolutely continuous with respect to λ. $\rho = \frac{d\mu}{d\lambda} : [0,1] \longrightarrow [0,\infty)$.

$$\rho\lambda(A) = \int_A \rho \, d\lambda.$$

$$\begin{aligned} \rho\lambda(T^{-1}(A)) &= \sum_{i=1}^{3} \rho\lambda(T_i^{-1}(A)) = \sum_{i=1}^{3} \int_{T_i^{-1}(A)} \rho \, d\lambda \\ &= \sum_{i=1}^{3} \int_A \rho \circ T_i^{-1}(x) |(T_i^{-1})'(x)| \, d\lambda(x) \\ &= \int_A \left[\sum_{i=1}^{3} \rho \circ T_i^{-1}(x) |(T_i^{-1})'(x)| \right] d\lambda(x) \end{aligned}$$

$$\lambda(A) = 0 \;\; \Rightarrow \;\; \rho\lambda(T^{-1}(A)) = 0.$$

Thus,

$$\mu \circ T^{-1} \preceq \lambda.$$

Quasi–Invariant Measures

Observation

$\mu \longmapsto \mu \circ T^{-1}$ preserves the space of probability measures absolutely continuous wrt λ.

$$\mathcal{L}_\lambda(\rho)(x) := \frac{d}{d\lambda}\rho\lambda \circ T^{-1}(x) = \sum_{i=1}^{3} \rho \circ T_i^{-1}(x)|(T_i^{-1})'(x)|$$
$$= \sum_{y \in T^{-1}(x)} \rho(y)|T'(y)|^{-1}$$

$\rho\lambda$ is T–invariant if and only if

$$\mathcal{L}_\lambda(\rho) = \rho.$$

Problem of finding fixed points of $\mathcal{L}_\lambda$.

Quasi–Invariant Measures

Generally: $T : X \to X$. A probability measure m on X is called quasi–invariant if and only if

$$m \circ T^{-1} \preceq m.$$

$$\mathcal{L}_m(\rho) := \frac{d}{dm}\rho m \circ T^{-1}, \quad \mathcal{L}_m : L^1(m) \to L^1(m)$$

ρm is T–invariant iff $\mathcal{L}_m \rho = \rho$

$$\int_X \mathcal{L}_m \rho \, dm = \int_X \rho \, dm \quad - \quad \text{also a defining property}$$

$\mathcal{L}_m$ is called the transfer operator of T wrt m.

Quasi–invariant measures abound. All piecewise differentiable maps on Riemannian manifolds with respect to their Riemannian volumes.

Smooth Expanding Maps

Definition

M – compact Riemannian manifold. $T : M \to M$ a C^2–differentiable map.
T is called (infinitesimally) expanding if and only if

$\exists k \geq 1 \; \exists \gamma > 1 \; \forall x \in M \; \forall v \in T_x M$

$$\|D_x T^k(v)\|_{T^k(x)} \geq \gamma \|v\|_x.$$

Examples: $S^1 \ni z \mapsto z^d$ ($|d| \geq 2$),
Toral expanding endomorphisms: ($k \in \mathbb{N}$),

$$A : \mathbb{R}^k \to \mathbb{R}^k$$

linear, with integral entries and all eigenvalues with moduli larger than 1.

$$\tilde{A} : \mathbb{R}^k/\mathbb{Z}^k \longrightarrow \mathbb{R}^k/\mathbb{Z}^k \text{ - smooth expanding map}$$

Also, all their sufficiently small C^2–perturbations.

Smooth Expanding Maps

λ – the normalized Riemannian volume on M; quasi–invariant.

$$\mathcal{L}_\lambda g(z) = \sum_{w \in T^{-1}(z)} g(w) |\det(D_w T)|^{-1}$$

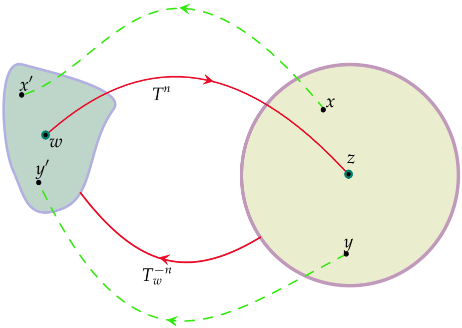


Figure: $B(x,\delta), T^n(x') = x, T^n(y') = y$ and $\frac{|\det(D_{y'}T^n)|^{-1}}{|\det(D_{x'}T^n)|^{-1}} \leq \exp(C d_M(x,y))$.

Smooth Expanding Maps

$$\frac{\mathcal{L}^n \mathbb{1}(y)}{\mathcal{L}^n \mathbb{1}(x)} \leq \exp\big(Cd_M(x,y)\big) \leq Q := \exp\big(C\,\mathrm{diam}(M)\big)$$

$$\int_M \mathcal{L}_\lambda^n \mathbb{1} d\lambda = \int_M \mathbb{1} d\lambda = 1 \implies \exists z_n \in M \;\; \mathcal{L}_\lambda^n \mathbb{1}(z_n) \leq 1$$

So, $\forall x \in M$: (uniform boundedness)

$$\mathcal{L}_\lambda^n \mathbb{1}(x) \leq Q.$$

Therefore,

$$\begin{aligned} |\mathcal{L}_\lambda^n \mathbb{1}(y) - \mathcal{L}_\lambda^n \mathbb{1}(x)| &\leq \mathcal{L}_\lambda^n \mathbb{1}(x)[\exp\big(Cd_M(x,y)\big) - 1] \\ &\leq Q[\exp\big(Cd_M(x,y)\big) - 1] \\ &\leq Dd_M(x,y) \end{aligned}$$

Smooth Expanding Maps

$$\left(\mathcal{L}_\lambda^n \mathbb{1}\right)_{n=1}^{\infty} \quad \text{is uniformly bounded and equicontinuous}$$

Thus,

$$\left(\frac{1}{n}\sum_{j=0}^{n-1} \mathcal{L}_\lambda^j \mathbb{1}\right)_{n=1}^{\infty} \quad \text{is uniformly bounded and equicontinuous}$$

$$\rho := \lim_{k\to\infty} \frac{1}{n_k} \sum_{j=0}^{n_k-1} \mathcal{L}_\lambda^j \mathbb{1}$$

satisfies

$$\mathcal{L}_\lambda \rho = \rho$$

So, [K. Krzyżewski, W. Szlenk, Studia Math. 1969]:

$\rho\lambda$ is T–invariant and absolutely continuous wrt λ

Smooth Expanding Maps

More:

The transfer operator $\mathcal{L}_\lambda : C(M) \to C(M)$ is almost periodic

meaning that for every $g \in C(M)$ the sequence

$$\left(\mathcal{L}_\lambda^n g\right)_{n=1}^{\infty} \subset C(M) \text{ is pre–compact}$$

Then

$$C(M) = C_u(M) \oplus C_0(M)$$

$C_u(M)$ is the closure of the linear span of eigenvectors with unitary eigenvalues and

$$C_0(M) = \left\{g \in C(M) : \lim_{n\to\infty} \mathcal{L}_\lambda^n(g) = 0\right\}.$$

$$\mathcal{L}_\lambda(C_u(M)) \subset C_u(M),\ \ \mathcal{L}_\lambda(C_0(M)) \subset C_0(M)$$

$$\mathcal{L}_\lambda = Q_1 + S,$$

$$Q_1 = \mathcal{L}_\lambda p_u,\ \ S = \mathcal{L}_\lambda p_0,\ \ Q_1 S = S Q_1 = 0,$$

Smooth Expanding Maps

In our expanding case:

$$C_u(M) = \mathbb{C}\rho$$

$$C_0(M) = \left\{ g \in C(M) : \int_M g \, d\lambda = 0 \right\}.$$

$$\mathcal{L}_\lambda = Q_1 + S, \quad \mathcal{L}_\lambda^n = Q_1 + S^n$$

$$Q_1 S = S Q_1 = 0$$

where the projector $Q_1 : C(M) \to \mathbb{C}\rho$, the eigenspace of the eigenvalue 1, is given by:

$$Q_1(g) = \left(\int g \, d\lambda \right) \rho,$$

and

$$\lim_{n\to\infty} S^n(g) = 0 \quad \forall g \in C(M).$$

Smooth Expanding Maps

The first ergodic consequence is mixing: $\forall f, g \in L^1(\rho\lambda)$ we have that

$$\lim_{n\to\infty} \int_M f \circ T^n \cdot g \, d\lambda = \int_M f \, d\lambda \cdot \int_M g \, d\lambda$$

Proof. WLG $\rho\lambda(f) = \rho\lambda(g) = 0$. Assume that f and g are continuous.

$$\begin{aligned}
\rho\lambda\big(f \circ T^n \cdot g\big) &= \lambda\big(f \circ T^n \cdot g\rho\big) = \lambda\big(\mathcal{L}_\lambda^n\big(f \circ T^n \cdot g\rho\big)\big) \\
&= \lambda\big(\mathcal{L}_\lambda^n\big(f \circ T^n \cdot g\rho\big)\big) = \lambda\big(f \mathcal{L}_\lambda^n(g\rho_\phi)\big) \\
&= \lambda\big(fS^n(g\rho)\big) \xrightarrow[n\to\infty]{} 0
\end{aligned}$$

Approximate L^1 functions by continuous ones.

Lasota–Yorke Maps

Definition

A map $T : I \to I$ is called Lasota–Yorke if

(a) T is piecewise expanding: $\exists$ a finite interval partition $\mathcal{P}$ of I and $\exists$ a constant $\gamma > 1$ s.t.

- $T|_J$ is differentiable for every $J \in \mathcal{P}$ and
- $$|T'(x)| \geq \gamma$$
  for all $x \in J$ and all $J \in \mathcal{P}$.

(b) $$1/|T'| \in \mathrm{BV}(I)$$
meaning that $1/|T'|$ is a function of bounded variation

M. Rychlik [Studia Math. 1983] allows countable partitions $\mathcal{P}$.

Lasota–Yorke Maps

Examples:

- β–transformations; $\beta > 1$,

$$I \ni x \longmapsto \beta x \pmod 1$$

- The Gauss map: $T : I \to I$,

$$T(x) := \frac{1}{x} \pmod 1.$$

Lasota–Yorke Maps

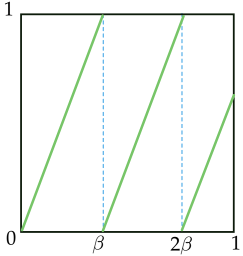


Figure: $\beta x (\text{mod } 1)$

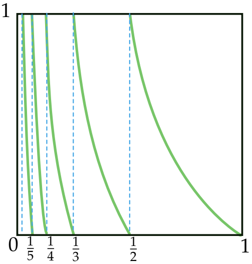


Figure: $1/x (\text{mod } 1)$

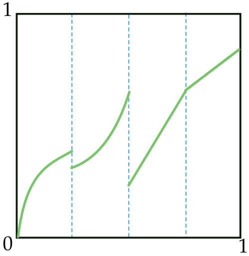


Figure: A BV map

Lasota–Yorke Maps

Lebesgue measure λ is quasi–invariant.

$$\|g\|_{\mathrm{BV}} := \|g\|_{L^1(\lambda)} + V(g)$$

$$\sup\{\|\mathcal{L}_\lambda^n \mathbb{1}\|_{\mathrm{BV}} : n \in \mathbb{N}\} < \infty$$

$$\overline{B}_{\mathrm{BV}}(0,1) \subset L^1(\lambda) \text{ is compact}$$

$$\rho := \lim_{k\to\infty} \frac{1}{n_k} \sum_{j=0}^{n_k-1} \mathcal{L}_\lambda^j \mathbb{1}$$

satisfies

$$\mathcal{L}_\lambda \rho = \rho$$

So, [A. Lasota, J. Yorke, Bull. Acad. Polon. Sci. 1977]:

$\rho\lambda$ is T–invariant and absolutely continuous wrt λ

$\mathcal{L}_\lambda : L^1(\lambda) \to L^1(\lambda)$ is almost periodic.

M. Rychlik [Studia Math. 1983]: Countable infinite partitions

Kolmogorov–Sinai Entropy

(X, d)–separable metric space. $T : X \to X$–Borel measurable map.
$n \in \mathbb{N}$:

$$d_n(x, y) := \max \left\{ d(T^k(x), T^k(y)) : k = 0, 1, \ldots, n-1 \right\}$$

$B_n(x, r) := B_{d_n}(x, r) =$

$$= \{ y \in X : d(y, x) < r,\ d(T(y), T(x)) < r, \ldots, d(T^{n-1}(y), T^{n-1}(x)) < r \}$$

μ–T–invariant Borel probability measure. Then for μ-almost every $x \in X$:

$$h_\mu(T) = \lim_{\delta \to 0} \liminf_{n \to \infty} \frac{-\log \mu(B_n(x, \delta))}{n} = \lim_{\delta \to 0} \limsup_{n \to \infty} \frac{-\log \mu(B_n(x, \delta))}{n},$$

This is the Kolmogorov–Sinai Entropy of T wrt μ.

$$\mu(B_n(x, \delta)) \sim e^{-h_\mu(T) n}$$

Topological Pressure and Variational Principle

If (X, d) is compact and $T : X \to X$ is continuous, then Borel probability measures always exist (Bogolubov-Krylov), even ergodic.

But one wants measures that capture some significant features of the system T. D. Ruelle, motivated by his research in statistical physics, suggested in late 1960s to look at the quantity ($\phi : X \to \mathbb{R}$ is a continuous function)

$$\sup \left\{ h_\mu(T) + \int_X \phi \, d\mu \right\},$$

where the supremum is taken over all $\mu \in M(T)$.

$$\mathrm{P}(\phi) := \lim_{\varepsilon \to 0} \frac{1}{n} \log \sum_{x \in E_n(\varepsilon))} \exp(S_n \phi(x)),$$

where $(E_n(\varepsilon))_{n=1}^{\infty}$ is any sequence of maximal ε-separated sets in X wrt d_n.

$$S_n \phi = \sum_{k=0}^{n-1} \phi \circ T^k.$$

Topological Pressure and Variational Principle

Throughout late 1960s–late 1970s the following was proved:

Theorem (Variational Principle; Ruelle, Walters, Bowen, Misiurewicz)

$$\mathrm{P}(\phi) = \sup \left\{ h_\mu(T) + \int_X \phi \, d\mu \right\}$$

Question: Do there exist maximizing measures? If so, are these unique?
Maximizing measures are called equilibrium states for ϕ.
How to construct them?

Ruelle's Operator and Conformal Measures

$T : X \to X$–continuous, open, surjective, locally 1–to–1.
$\mathcal{L}_\phi : C(X) \to C(X)$ – the associated Ruelle (Perron–Frobenius, transfer) operator.

$$\mathcal{L}_\phi g(x) := \sum_{y \in T^{-1}(x)} g(y) e^{\phi(y)}.$$

The dual operator: $\mathcal{L}_\phi^* : C^*(X) \longrightarrow C^*(X)$

$$\mathcal{L}_\phi^* \nu(g) = \nu(\mathcal{L}_\phi g).$$

$\Phi : M(X) \to M(X)$:

$$\Phi(\nu) := \frac{\mathcal{L}_\phi^* \nu}{\mathcal{L}_\phi^* \nu(\mathbb{1})}$$

$C^*(X)$ is locally convex, $M(X)$ is convex and compact, Φ is continuous, so by the Schauder–Tichonov Theorem, Φ has a fixed point m.

$$\mathcal{L}_\phi^* m = \lambda m,$$

where $\lambda = \mathcal{L}_\phi^* m(\mathbb{1}) > 0$.

Ruelle's Operator and Conformal Measures

Equivalently:

$$m(T(A)) = \lambda \int_A e^{-\phi}\, dm = \int_A \exp(\log\lambda - \phi)\, dm.$$

whenever $A \subset X$ is Borel and $T|_A$ is 1–to–1. m is then referred to [M. Denker, M.U., 1989] $\exp(\log\lambda - \phi)$–conformal.

m is quasi–invariant

$$\mathcal{L}^*_{\phi - \log\lambda} m = m.$$

We want an invariant measure absolutely continuous wrt m; a candidate for an equilibrium state for ϕ.

Question: Under which circumstances is the sequence

$$\mathcal{L}^n_{\phi - \log\lambda} \mathbb{1}, \ \ n \in \mathbb{N},$$

uniformly bounded and equicontinuous?

Distance Expanding Maps

Definition (D. Ruelle A-W (1978), cf. F. Przytycki, M. U., CUP (2010))

A continuous map $T : X \to X$ of a compact metric space (X, d) is called distance expanding provided that there exist two constants $\lambda > 1$ and $\delta > 0$ such that

$$d(x, y) < 2\delta \implies d\big(T(x), T(y)\big) \geq \lambda\, d(x, y).$$

We want T to be also open and topologically transitive. Smooth expanding maps are such, examples abound:

Distance Expanding Maps; SFT

Subshifts of finite type:
E–a finite set. $E^{\mathbb{N}}$.

$$d_2(\omega,\tau) = 2^{-k},$$

where k is the length of the longest common initial block of $\omega = \omega_1\omega_2\ldots$ and $\tau = \tau_1\tau_2\ldots$.

$$\sigma : E^{\mathbb{N}} \to E^{\mathbb{N}}, \;\; \sigma(\omega_1\omega_2\omega_3\ldots) = \omega_2\omega_3\omega_4\ldots$$

The shift map.
$d_2(\omega,\tau) < 1 \Rightarrow \omega_1 = \omega_2$. Thus,

$$d_2(\sigma(\omega),\sigma(\tau)) = 2d_2(\omega,\tau) - \text{expanding}$$

$A : E \times E \longrightarrow \{0,1\}$–incidence matrix.

$$E_A^\infty := \left\{\omega \in E^\infty : A_{\omega_n\omega_{n+1}} = 1, \forall\, n \in \mathbb{N}\right\}.$$

$$\sigma(E_A^\infty) \subset E_A^\infty, \;\; \sigma : E_A^\infty \longrightarrow E_A^\infty \text{ open}$$

Transitive iff the matrix A is irreducible.

Distance Expanding Maps; Expanding Cantor Sets

1. E–a finite set with at least 2 elements.
2. $X_0 \subset \mathbb{R}^d$–a compact set. For example, a closed cube or ball.
3. $\phi_e : X_0 \to X_0, e \in E,$ – (strict) injective contractions.
4. $\varphi_a(X_0) \cap \varphi_b(X_0) = \emptyset$ for all $a, b \in E$ with $a \neq b$.

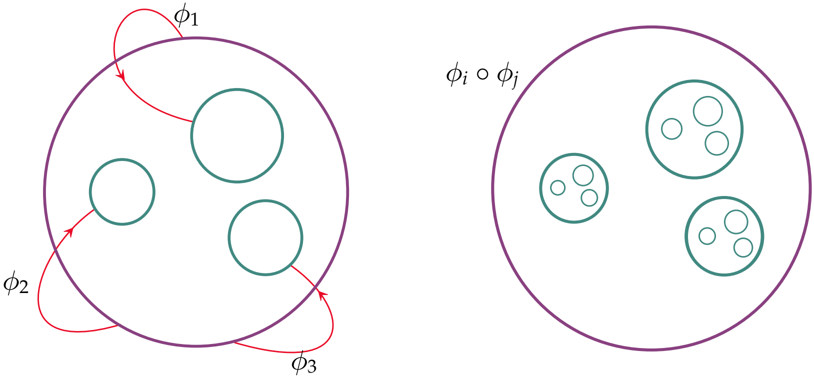
$$X_n := \bigcup_{(\omega_1,\omega_2,\dots,\omega_n)\in E^n} \varphi_{\omega_1} \circ \varphi_{\omega_2} \circ \cdots \circ \varphi_n(X_0)$$

$$X := \bigcap_{n=1}^{\infty} X_n, \text{ non} - \text{empty}, \text{ compact}, \text{ Cantor}$$

$$T : X \to X, \ T(\phi_e(x)) = x \ - \ \text{expanding}, \text{ open}, \text{ transitive}$$

Generalizations: Expanding repellers, ex. expanding rational functions, ex. $z \mapsto z^2 + c$, $|c| << 1$, iterated function systems, graph directed Marcov systems, countable alphabet E.

Distance Expanding Maps; Expanding Cantor Sets



Iterated Function Systems

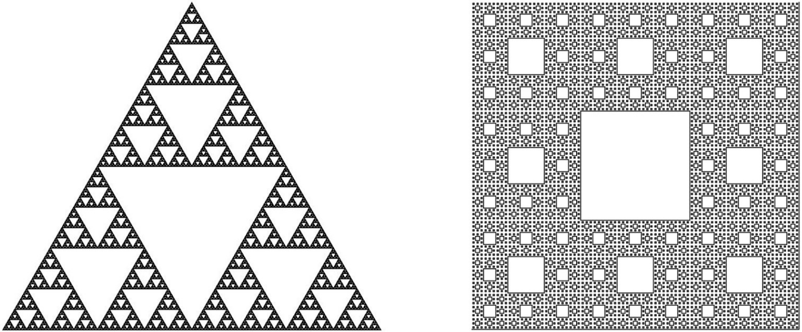


Figure: Sierpiński Triangle and Carpet

Distance Expanding Maps and Hölder Potentials; [D. Ruelle (AW-1978)], [F. Przytycki, M. U. (CUP-2010)]

(X, d) – a compact metric space. $\varphi : X \to \mathbb{R}$ is Hölder continuous with exponent $\alpha > 0$ iff there exists some $C \geq 0$ such that

$$|\varphi(x) - \varphi(y)| \leq C\, d^{\alpha}(x, y), \quad \forall\, x, y \in X.$$

$v_\alpha(\varphi)$ – the least such C.
$\mathrm{H}_\alpha(X)$

$$\|\varphi\|_\alpha := v_\alpha(\varphi) + \|\varphi\|_\infty.$$

Then $(\mathrm{H}_\alpha(X), \|\cdot\|_\alpha)$ is a Banach space.

$T : X \to X$ – transitive, open, distance expanding. The linear operators

$$\mathcal{L}_\phi(C(X)) \subset C(X), \;\; \mathcal{L}_\phi : C(X) \longrightarrow C(X)$$

and

$$\mathcal{L}_\phi(\mathrm{H}_\alpha(X)) \subset \mathrm{H}_\alpha(X), \;\; \mathcal{L}_\phi : \mathrm{H}_\alpha(X) \longrightarrow \mathrm{H}_\alpha(X)$$

are bounded.

Distance Expanding Maps and Hölder Potentials; [D. Ruelle (AW-1978)], [F. Przytycki, M. U. (CUP-2010)]

$$\lambda = e^{\mathrm{P}(\phi)} = r(\mathcal{L}_\phi)$$

$$\hat{\phi} := \phi - \mathrm{P}(\phi).$$

$$\hat{\mathcal{L}}_\phi := \mathcal{L}_{\hat{\phi}}$$

The transfer operator $\hat{\mathcal{L}}_\phi : C(X) \longrightarrow C(X)$ is almost periodic

$$\hat{\mathcal{L}}_\phi^* m_\phi = m_\phi, \ \ \hat{\mathcal{L}}_\phi \rho_\phi = \rho_\phi, \ \ \rho_\phi - \text{Hölder}.$$

m_ϕ quasi–invariant. $\mu_\phi = \rho_\phi m_\phi$–a unique equilibrium state of ϕ.

Rational Functions on the Riemann Sphere

**Theorem (Several papers:
F. Przytycki, Bol. Soc. Bras. Mat. (1990),
M. Denker, M.U., Nonlinearity (1991),
M. Denker, F. Przytycki, M.U., Ergod. Th. & Dynam. Sys. (1996),
cf. S. Munday, M. Roy, M.U., De Gruyter (2023))**

If $T : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is a rational function, $\varphi : J(T) \to \mathbb{R}$ is Hölder continuous, and

$$\mathrm{P}(\phi) > \sup(\phi) \ \ (\text{implied by} \sup(\phi) - \inf(\phi) < h_{top}(T) = \log \deg(T)),$$

then

The Ruelle operator $\hat{\mathcal{L}}_\phi : C(J(T)) \longrightarrow C(J(T))$ is almost periodic

$$\hat{\mathcal{L}}_\phi^* m_\phi = m_\phi, \ \ \hat{\mathcal{L}}_\phi \rho_\phi = \rho_\phi, \ \ \rho_\phi - \text{Hölder}.$$

m_ϕ quasi–invariant. $\mu_\phi = \rho_\phi m_\phi$–a unique equilibrium state of ϕ.

Note: If critical points are in $J(T)$, then $\mathcal{L}_\phi(\mathrm{H}_\alpha(J(T))) \not\subset \mathrm{H}_\alpha(J(T))$.

Julia Sets for Rational and Meromorphic Functions

Definition

A point $z \in \widehat{\mathbb{C}}$ belongs, by definition, to the Fatou set $F(T)$ of T if and only if there exists an open neighborhood $U \subset \widehat{\mathbb{C}}$ of z such that the family of iterates

$$T^n|_U : U \longrightarrow \widehat{\mathbb{C}}, \ \ n \geq 1,$$

is equicontinuous.

The Julia set $J(T)$ of T is defined to be

$$J(T) = \widehat{\mathbb{C}} \setminus F(T).$$

1. $F(T)$ is open and $J(T)$ is closed.
2. $J(T) \neq \emptyset$.
3. $J(T)$ is perfect.
4. Either $J(T)$ is nowhere dense or $J(T) = \widehat{\mathbb{C}}$.
5. $T(J(T)) = J(T) = T^{-1}(J(T))$

We can consider the dynamical system

$$T : J(T) \longrightarrow J(T)$$

Rational Julia Sets

$z \mapsto z^2$; $z \mapsto z^2 + c, 0 < |c| << 1$; $z \mapsto z^2 - 2$; $z \mapsto z^2 + i$;
$z \mapsto z^2 + (1/4)$

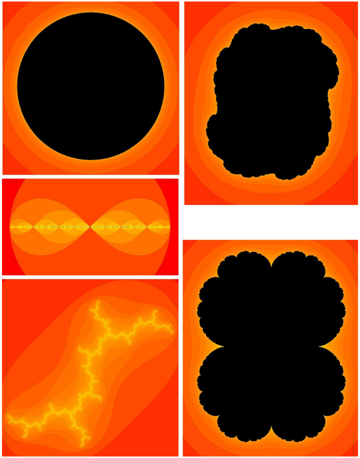


Figure: Rational Julia Sets

Exponential Julia Set

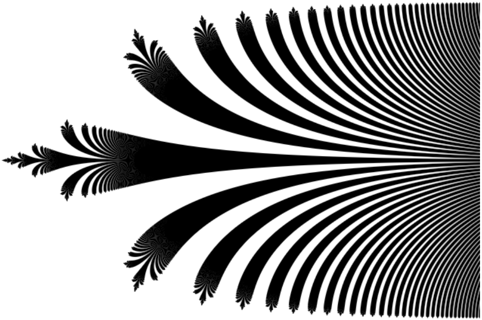


Figure: Exponential Function $z \longmapsto \lambda e^z, \ \ \lambda \in (0, 1/e)$

Exponential Julia Set

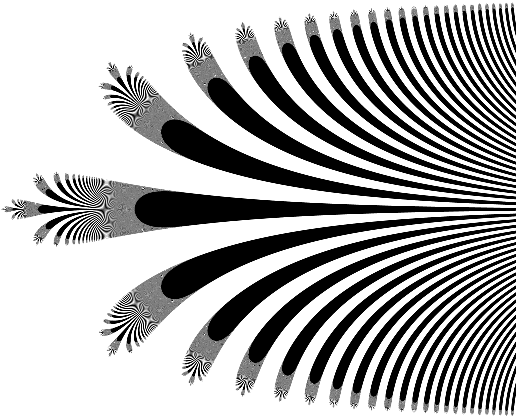


Figure: Exponential Function $z \longmapsto \lambda e^z, \ \ \lambda \in (0, 1/e)$

Exponential Julia Set

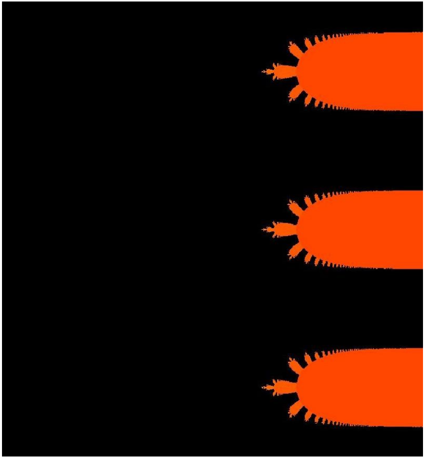
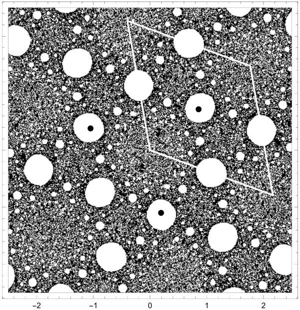
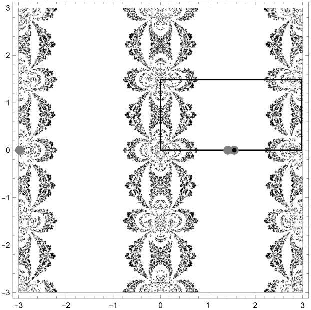
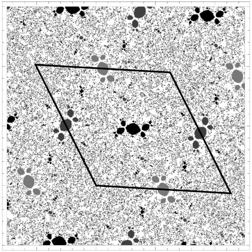


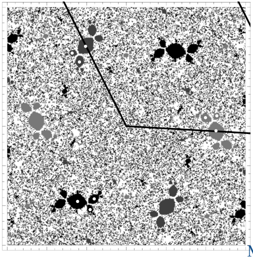
Figure: Exponential Function $z \longmapsto \lambda e^z, \quad \lambda \in (0, 1/e)$

Julia sets of Weierstrass $\wp$-elliptic functions









Holomorphic Endomorphisms of Complex Projective Spaces

The Julia set $J(T)$ is defined to be the topological support of the unique measure of maximal entropy.

Theorem (A. Zdunik, M.U., Fund. Math. (2013))

If $T : \mathbb{CP}^k \to \mathbb{CP}^k$ is a holomorphic non–exceptional endomorphism, $\varphi : J(T) \to \mathbb{R}$ is Hölder continuous, and $\sup(\phi) - \inf(\phi)$ is sufficiently small, then

The transfer operator $\hat{\mathcal{L}}_\phi : C(J(T)) \longrightarrow C(J(T))$ is almost periodic

$$\hat{\mathcal{L}}_\phi^* m_\phi = m_\phi, \;\; \hat{\mathcal{L}}_\phi \rho_\phi = \rho_\phi, \;\; \rho_\phi - \text{Hölder.}$$

m_ϕ quasi–invariant. $\mu_\phi = \rho_\phi m_\phi$–a unique equilibrium state of ϕ.

Countable Shift

E–countable, possibly infinite.

$$\sigma : E_A^\infty \longrightarrow E_A^\infty$$

$\phi : E_A^\infty \longrightarrow \mathbb{R}$ is

1. Hölder continuous, with a uniform Hölder variation, on each cylinder
$$[e] := \{\omega \in E_A^\infty : \omega_1 = e\}.$$
2. summable:
$$\sum_{e \in E} \exp\left(\sup(\phi|_{[e]]})\right) < +\infty.$$

The Ruelle operator:

$$\mathcal{L}_\phi(g)(\omega) := \sum_{\substack{e \in E \\ A_{e\omega_1}=1}} g(e\omega) \exp\left(\phi(e\omega)\right).$$

Countable Shift

The following two operators are bounded:

$$\mathcal{L}_\phi(C_b(E_A^\infty)) \subset C_b(E_A^\infty), \;\; \mathcal{L}_\phi : C_b(E_A^\infty) \longrightarrow C_b(E_A^\infty)$$

and

$$\mathcal{L}_\phi(\mathrm{H}_\alpha(E_A^\infty)) \subset \mathrm{H}_\alpha(E_A^\infty), \;\; \mathcal{L}_\phi : \mathrm{H}_\alpha(E_A^\infty) \longrightarrow \mathrm{H}_\alpha(E_A^\infty)$$

$$\hat{\mathcal{L}}_\phi := \phi - \log(r(\mathcal{L}_\phi))$$

Theorem (D. Mauldin, M.U., Israel J. (2001), CUP (2003))

If $A^p > 0$ for some integer $p \geq 1$ (finitely primitive matrix), then

$$\hat{\mathcal{L}}_\phi^* m_\phi = m_\phi, \;\; \hat{\mathcal{L}}_\phi \rho_\phi = \rho_\phi, \;\; \rho_\phi - \text{Hölder}.$$

m_ϕ quasi–invariant. $\mu_\phi = \rho_\phi m_\phi$–a unique equilibrium state of ϕ.

$\mathcal{L}_\phi : C_b(E_A^\infty) \longrightarrow C_b(E_A^\infty)$ is almost periodic.

Transcendental Meromorphic Functions

[K. Barański, Fund. Mat. (1995)] $\mathbb{C} \ni z \longmapsto \lambda \tan(z)$, $|\lambda| > 1$, and more.
[J. Kotus, M.U., Math. Annalen (2002)] $H \circ \exp \circ Q : \mathbb{C} \longrightarrow \widehat{\mathbb{C}}$,
[A. Zdunik, M.U., Ergod. Th. & Dynam. Sys. (2004)], expanding exponential functions,
[J. Kotus, M.U., Discrete & Continuous Dyn. Sys. 13 (2005)], Fatou functions,
[V. Mayer, M.U., Ergod. Th. & Dynam. Sys. (2008), Memoirs AMS (2010], large classes of meromorphic functions
[V. Mayer, M.U., Transactions AMS (2020)], large classes of functions $f : \mathbb{C} \to \widehat{\mathbb{C}}$ meromorphic, expanding on its Julia set. $t > 0$ large enough,

$$\phi_t := -t \log |f'| : J(f) \longrightarrow \mathbb{R}$$

The Ruelle operator:

$$\mathcal{L}_t = \mathcal{L}_{\phi_t} : C_b(J(f)) \longrightarrow C_b(J(f))$$

$$\mathcal{L}_t(g)(z) := \sum_{w \in f^{-1}(z)} g(w)|f'(w)|^{-t}$$

Transcendental Meromorphic Functions: Problem

$$f_\lambda(z) = \lambda e^z$$

$\lambda \in \mathbb{C} \setminus \{0\}$:

$$\mathcal{L}_t(\mathbb{1})(z) = \sum_{w \in f_\lambda^{-1}(z)} |f_\lambda'(w)|^{-t} = \sum_{w \in f_\lambda^{-1}(z)} |z|^{-t} = +\infty,$$

always

Transcendental Meromorphic Functions: Remedies

Conformal change of Riemannian metric on $\mathbb{C}$ ([V. Mayer, M.U., 2008, 2010, 2020]):

$$|dz|/|z|.$$

Then

$$|f'(z)|_1 = |f'(z)|\frac{|z|}{|f(z)|}.$$

So,

$$|f_\lambda'(z)|_1 = |z|$$

Therefore

$$\mathcal{L}_t \mathbb{1}(w) = \sum_{z\in f_\lambda^{-1}(w)} |f'(z)|_1^{-t} = \sum_{z\in f_\lambda^{-1}(w)} |z|^{-t} = \sum_{n\in\mathbb{Z}} |\log(w/\lambda) + 2\pi i n|^{-t}$$

works for $t > 1$.

Transcendental Meromorphic Functions

$$d\sigma(z) = |dz|/|z|^{\beta}, \ \ \phi_t := -t \log |f'|_\sigma$$

Theorem (Zdunik, M.U., (2004); J. Kotus, M.U., 2005; V. Mayer, M.U., (2008), (2010), (2020))

For a large class of expanding transcendental meromorphic functions and appropriately large $t > 0$:

$$\mathcal{L}_t(\mathrm{H}_\alpha) \subset \mathrm{H}_\alpha$$

$$\hat{\mathcal{L}}_t^* m_t = m_t, \ \ \hat{\mathcal{L}}_t \rho_t = \rho_t, \ \ \rho_t - \text{Hölder}.$$

m_t *quasi–invariant*. $\mu_t = \rho_t m_t$*–a unique equilibrium state of* ϕ_t.

Use of Koebe's Distortion theorems and Nevanlina Theory to control the growth of the operators $\mathcal{L}_t$.

Quasi–Compactness and Spectrum Gap

We are interested in stochastic properties of the above dynamical systems (T, μ_ϕ). They come from spectral properties of $\mathcal{L}_\phi$. Except for rational functions and complex projective spaces, we have this.

Theorem (Spectral Gap)

(a) *The spectral radius* $r(\hat{\mathcal{L}}_\phi) = 1$*.*

(b) *The number* 1 *is a simple isolated eigenvalue of the operator*

$$\hat{\mathcal{L}}_\phi : \mathrm{H}_\alpha \to \mathrm{H}_\alpha$$

and the rest of the spectrum is contained in a disk of radius strictly smaller than 1 *(more than quasi–compactness). More precisely:*

(c) *There exists a bounded linear operator* $S : \mathrm{H}_\alpha \to \mathrm{H}_\alpha$ *such that*

$$\hat{\mathcal{L}}_\phi = Q_1 + S, \ \ Q_1 S = S Q_1 = 0,$$

where the projector $Q_1 : \mathrm{H}_\alpha \to \mathbb{C}\rho_\phi$*, the eigenspace of* 1*, is:*

$$Q_1(g) = \left(\int g \, dm_\phi \right) \rho_\phi, \ \ \|S^n\|_\alpha \leq C\xi^n, \quad \xi \in (0,1), \ \forall n \geq 1$$

Quasi–Compactness and Spectral Gap

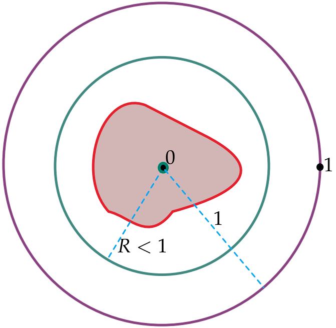


Figure: $\hat{\mathcal{L}}_\phi = Q_1 + S$; $\hat{\mathcal{L}}_\phi^n = Q_1 + S^n$

Complex Projective Spaces

$T : \mathbb{CP}^k \to \mathbb{CP}^k$ is a holomorphic "non–degenerate" endomorphism, $\varphi : J(T) \to \mathbb{R}$ is Hölder continuous, and $\sup(\phi) - \inf(\phi)$ is sufficiently small.

Spectral Gap: [F. Bianchi, T. C. Dinh, Journal de Mathématiques Pures et Appliqués (to appear); Preprint 2020]

More general potentials, non–standard Banach space, pluri–potential methods.

Stochastic Properties

The spectral gap entails the following.

Theorem

$f \in L^1(\mu_\phi)$ all $g \in B$.
Exponential Decay of Correlations:

$$\left| \int (f \circ T^n \cdot g)\, d\mu_\phi - \int f\, d\mu_\phi \int g\, d\mu_\phi \right| \le C\xi^n.$$

Central Limit Theorem: If f is not cohomological to a constant, then with some with some $\sigma > 0$,

$$\frac{\sum_{j=0}^{n-1} f \circ T^j - n \int f\, d\mu_t}{\sqrt{n}} \longrightarrow \mathcal{N}(0, \sigma^2)$$

Law of Iterated Logarithm: for μ_ϕ–a.e. $x \in X$:

$$\overline{\lim_{n\to\infty}} \frac{\sum_{j=0}^{n-1} f \circ T^j(x) - n \int f\, d\mu_\phi}{\sqrt{n \log \log n}} = \sqrt{2}\sigma.$$

Stochastic Properties

1. Exponential Decay of Correlations follows by a direct calculation from the Spectral Decomposition of $\mathcal{L}_\phi$:
   WLG: $\mu_\phi(f) = \mu_\phi(g) = 0$.
   $$\begin{aligned}\mu_\phi\big(f \circ T^n \cdot g\big) &= m_\phi\big(f \circ T^n \cdot g\rho_\phi\big) = m_\phi\Big(\mathcal{L}^n_{\hat{\phi}}\big(f \circ T^n \cdot g\rho_\phi\big)\Big) \\ &= m_\phi\Big(\mathcal{L}^n_{\hat{\phi}}\big(f \circ T^n \cdot g\rho_\phi\big)\Big) = m_\phi\big(f\mathcal{L}^n_{\hat{\phi}}(g\rho_\phi)\big) \\ &= m_\phi\big(fS^n(g\rho_\phi)\big) \le C\|f\|_{L^1(m_\phi)}\xi^n\end{aligned}$$
2. Central Limit Theorem follows by applying M. Gordin's result [Dokl. Akad. Nauk SSSR, (1969)] or a direct, though involved, calculation in [W. Parry, M. Pollicott, Astérisque (1990)] starting from the spectral decomposition of the Ruelle operator.
3. Almost Sure Invariance Principle [S. Gouezel, Annals of Prob. (2010)]: The sequence of random variables
   $$X \ni x \longmapsto S_nf(x) - n\mu_\phi(f), \ \ n \in \mathbb{N},$$
   can be approximated sufficiently well by a Brownian motion. ASIP entails both CLT and LIL.

Orbit Counting

Theorem (S. Lalley, Acta Math. (1989); M. Pollicott, M.U., Memoirs AMS (2021); O. Ivrii, M.U., in preparation)

E–countable set, $A : E \times E \longrightarrow \{0, 1\}$–finitely irreducible incidence matrix, and $\psi : E_A^\infty \to \mathbb{R}$, a D–generic Hölder summable potential such that $\mathrm{P}(\psi) = 0$ *and*

$$\int_{E_A^\infty} \psi^2 d\mu_\psi < +\infty.$$

Given $\xi \in E_A^\infty$ and a Borel set $B \subset E_A^\infty$ with $m_\psi(\partial B) = 0$, the counting function

$$N_\xi^B(T) := \#\{\omega \in E_\xi^* : \ \omega\xi \in B \text{ and } \exp(S_{|\omega|}(-\psi)(\omega\xi)) \le T\}$$

satisfies

$$\lim_{T\to\infty} \frac{N_\xi^B(T)}{T} = \frac{\rho_\psi(\xi)}{\int_{E_A^\infty} (-\psi) d\mu_\psi} \cdot m_\psi(B).$$

Orbit Counting; Idea of Proof

Assume that A is finitely primitive, $\sup(\psi) < 0$, and $u\psi$ is summable for some $u < 1$. Complexify the Ruelle operator:

$$\mathcal{L}_{s\psi} : \mathrm{H}_\alpha \longrightarrow \mathrm{H}_\alpha,\ s \in \mathbb{C},\ \mathrm{Re}(s) > u.$$

$$r(\mathcal{L}_{s\psi}) < 1$$

on some neighborhood of $\{\mathrm{Re}(s) = 1\} \setminus \{1\}$. Poincaré series:

$$\eta_\xi(s) := \sum_{n=0}^{\infty} \sum_{\omega \in E_A^n : A_{\omega_n \xi_1} = 1} \exp\big(S_n(s\psi)(\omega\xi)\big) = \sum_{n=0}^{\infty} \mathcal{L}_{s\psi}^n \mathbb{1}(\xi)$$

$$= \sum_{n=0}^{\infty} \left(\lambda_s^n Q_s + S_s^n\right) = (1-\lambda_s)^{-1} Q_s + \sum_{n=0}^{\infty} S_s^n,\ \ s \asymp 1.$$

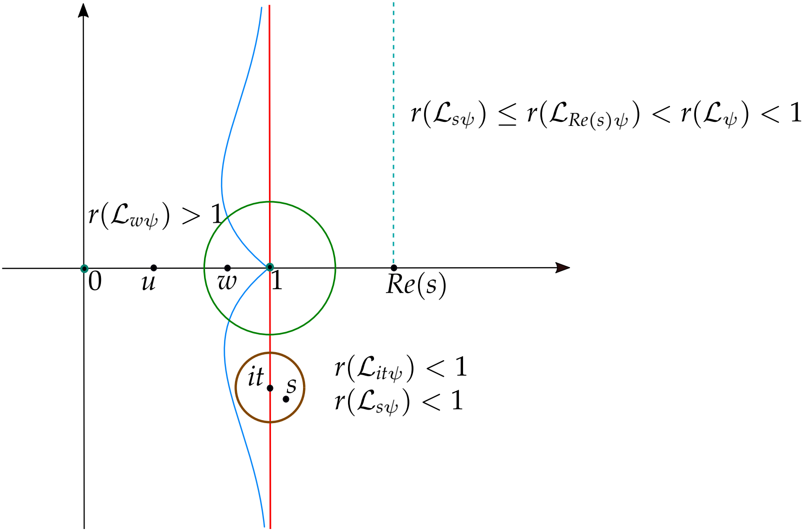
So, η_ξ has a meromorphic extension beyond $\mathrm{Re}(s) = 1$ with the only (simple) pole at $s = 1$ and we can calculate its residue:

$$\rho_\psi(\xi)/\mu_\psi(-\psi)$$

Observe that

$$\eta_\xi(s) = \int_0^\infty T^{-s} dN_\xi(T).$$

Orbit counting



Ikehara–Wiener Tauberian Theorem

and apply

Theorem (Ikehara–Wiener Tauberian Theorem)

Let M and θ be positive real numbers. Assume that

$$\alpha : [M, +\infty) \longrightarrow (0, +\infty)$$

is monotone increasing and continuous from the left, and also that there exists a (real) number $D > 0$ such that the function

$$s \longmapsto \int_M^{+\infty} x^{-s} d\alpha(x) - \frac{D}{s-\theta} \in \mathbb{C}$$

is analytic in a neighborhood of $\mathrm{Re}(s) \geq \theta$. Then

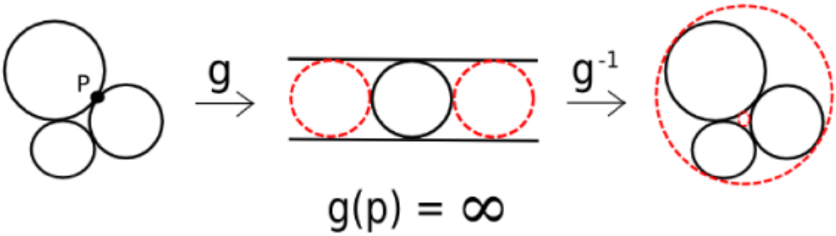
$$\lim_{x \to +\infty} \frac{\alpha(x)}{x^\theta} = \frac{D}{\theta}.$$

Orbit Counting: Apollonius Theorem

Theorem (Apollonius of Perga, 262–190 BC)

Given 3 mutually tangent circles, there exist exactly two circles tangent to all three.

Proof. $g : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ – a Möbius map



Apollonian Triangle

Theorem (A. V. Kontorovich, H. Oh, Journal AMS (2011); M. Pollicott, M.U., (2021))

1. C_1, C_2, C_3–*three mutually tangent circles in* $\mathbb{C}$ *with disjoint interiors.*
2. $\mathcal{T}$–*the curvilinear triangle formed by these circles.*
3. $C_{\mathcal{T}}$–*the circle packing of* $\mathcal{T}$
4. $\delta = 1.30561\ldots$–*the Hausdorff dimension of the residual set* $J_{\mathcal{T}}$ *of* $C_{\mathcal{T}}$.

If $N_{\mathcal{T}}$ *is the number of circles* S *in* $C_{\mathcal{T}}$ *with* $\operatorname{diam}(S) \geq 1/T$, *then the limit*

$$L = \lim_{T \to +\infty} \frac{N_{\mathcal{T}}(T)}{T^{\delta}}$$

exists, is positive, and finite.

Apollonian Triangle

Moreover, if

1. H_δ is the Hausdorff measure on $J_{\mathcal{T}}$. $0 < \mathrm{H}_\delta(J_{\mathcal{T}}) < +\infty$,
2. $B \subset \mathbb{C}$ is a Borel set with $m_\delta(\partial B) = 0$ and
3. $N_{\mathcal{T}}(T; B)$ is the number of circles S in $\mathcal{T}$ with

$$\mathrm{diam}(S) \geq 1/T \text{ that intersect } B,$$

then

$$\lim_{T\to+\infty} \frac{N_{\mathcal{T}}(T;B)}{T^\delta} = \frac{L}{\mathrm{H}_\delta(J_{\mathcal{T}})} \mathrm{H}_\delta(B).$$

Apollonian Gasket

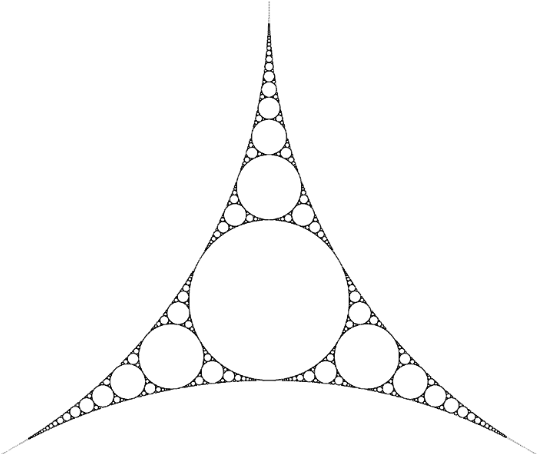


Figure: The Apollonian Gasket

Apollonian Gasket

Strategy of the proof:

1. Associate to each conformal graph directed Markov system (CGDMS) a corresponding countable shift and potential capturing the derivatives of contractions.
   Prove the corresponding orbit counting results for CGDMSs. More precisely, for diameters of the images of the seed set.
2. Associate to each (finite) parabolic conformal graph directed Markov system an ordinary (contracting) one. Its alphabet is always infinite. Get the orbit counting results for diameters of the images of the seed set.
3. Associate to the Apollonian triangle the corresponding parabolic GDMS (in fact IFS).

Apollonian Gasket

$C_1 = \phi_1(C_0), \;\; C_2 = \phi_2(C_0), \;\; C_3 = \phi_3(C_0)$

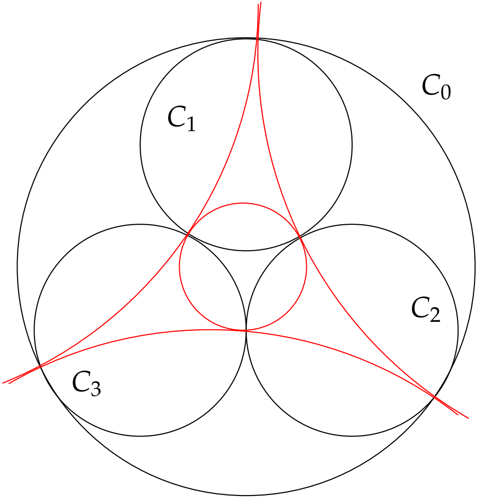
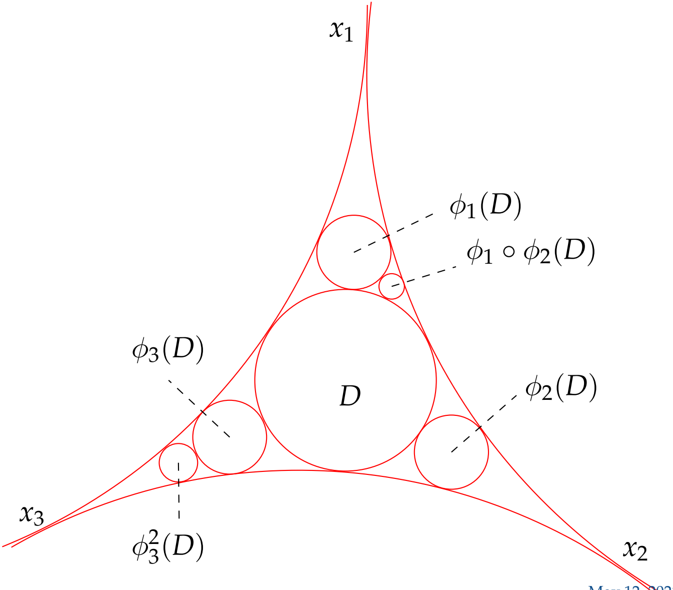


Figure: The Parabolic Iterated Function System; Möbius Maps

Apollonian Packing



Apollonian Packing

Theorem (A. V. Kontorovich, H. Oh, Journal AMS (2011); M. Pollicott, M.U., (2021))

1. *C_1, C_2, C_3–three mutually tangent circles in $\mathbb{C}$.*
2. *C_0-the circle tangent to all the circles C_1, C_2, C_3 and having all of them in its interior.*
3. *$\mathcal{A}$–the corresponding circle packing.*
4. *$\delta = 1.30561\ldots$–the Hausdorff dimension of the residual set $J_{\mathcal{A}}$ of $\mathcal{A}$.*
5. *$N_{\mathcal{A}}(T)$–the number of circles S in $\mathcal{A}$ with $\operatorname{diam}(S) \geq 1/T$.*

Then the limit

$$L = \lim_{T \to +\infty} \frac{N_{\mathcal{A}}(T)}{T^{\delta}}$$

exists, is positive, and finite.

Apollonian Packing

Moreover, if

1. H_δ is the Hausdorff measure on $J_{\mathcal{A}}$. $0 < \mathrm{H}_\delta(J_{\mathcal{A}}) < +\infty$,
2. $B \subset \mathbb{C}$ is a Borel set with $m_\delta(\partial B) = 0$ and
3. $N_{\mathcal{A}}(T; B)$ is the number of circles S in $\mathcal{T}$ with

$$\mathrm{diam}(S) \geq 1/T \text{ that intersect } B,$$

then

$$\lim_{T \to +\infty} \frac{N_{\mathcal{T}}(T; B)}{T^\delta} = \frac{L}{\mathrm{H}_\delta(J_{\mathcal{T}})} \mathrm{H}_\delta(B).$$

Apollonian Packing

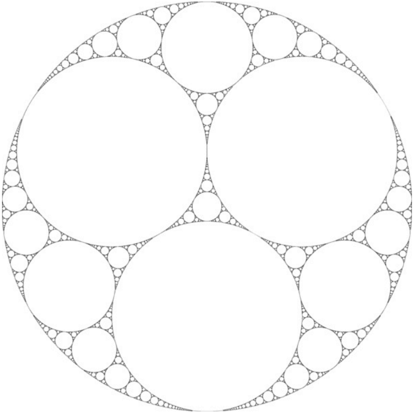


Figure: The Standard Apollonian Packing

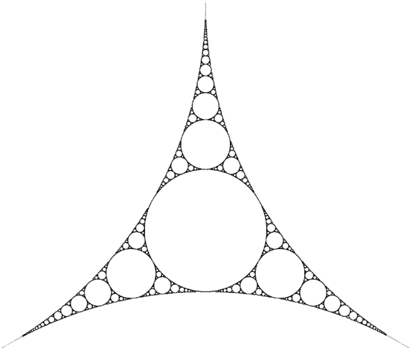


Figure: The Apollonian Gasket

Classical Schottky Groups

Remark: The strategy of the proof outlined for Apollonian gasket can be done more generally, namely for finitely generated Scottky groups (with tangencies or not).
The Apollonian packing is captured by a Schottky group.

For classical Schottky groups one can obtain with such dynamical approach counting results more specific to them and the hyperbolic spaces they induce:
For ex. the asymptotics related to the number of closed geodesics on the corresponding hyperbolic manifold whose lengths are bounded above by T.

Bowen's Formula

$T : X \to X$ open, expanding, transitive, and conformal.

Theorem (Bowen, Publ. Math. IHES, 1979)

If X is the limit set of a quasi–Fuchsian group, then

$$\mathrm{HD}(X) \textit{ is the only } t \in \mathbb{R} \textit{ such that } r\big(\mathcal{L}_{-t\log|T'|}\big) = 1.$$

Rational functions:

1. Expanding (D. Ruelle)
2. Expansive (no critical points in the Julia set), ex. $z \mapsto z^2 + (1/4)$ (M. Denker, M.U.)
3. With critical points in the Julia set (J. Graczyk, F. Przytycki, J. Rivera–Letelier, S. Smirnov, B. Stratmann, M.U.)
4. Rational Semigroups (J. Atnip, H. Sumi, M.U.)

Bowen's Formula

Conformal iterated function systems and graph directed Markov systems (P. Moran, J. Hutchinson, T. Bedford, D. Mauldin, M.U.)

Transcendental meromorphic functions (K. Barański, B. Karpińska, J. Kotus, V. Mayer, A. Zdunik, M.U.)

Bowen's Formula

Numerical estimates of HD(X) for conformal iterated function systems:

O. Jenkinson, C. McMullen, M. Pollicott, P. Vytnova – finite alphabet

V. Chousionis, S. Falk, S. Heinemann, D. Leykekhman, R. Nussbaum, M.U. – countable alphabet

For example: [O. Jenkinson, M. Pollicott Adv. Math., 325 (2018)]:

$$\mathrm{HD}(J_{\{1,2\}}) = 0.531280506277205141624468647 3\ldots$$

M. Pollicott and P. Vytnova got 200 decimals.
[V. Chousionis, D. Leykekhman, M.U Trans. AMS, (2020)]:

$$\mathrm{HD}(J_{\text{even}}) = 0.719\ldots,\ \mathrm{HD}(J_{\text{odd}}) = 0.821\ldots,\ \mathrm{HD}(J_{\text{prime}}) = 0.675\ldots$$

J_E is the set of all real numbers in $[0, 1]$ whose all continued fraction expansion entries are in E.
Applications to Markov and Lagrange spectra in Diophantine approximations.

Bowen's Formula

Real Analyticity of Hausdorff dimension:
Use of Kato–Rellich Perturbation Theorem based on Bowen's Formula.

1. Expanding rational functions (D. Ruelle)
2. Parabolic polynomials (H. Akter, M.U.)
3. Graph directed Markov systems (M. Roy, B. Skorulski, H. Sumi, M.U.)
4. Transcendental meromorphic functions (J. Kotus, V. Mayer, B. Skorulski, A. Zdunik, M.U.). Directly and with the use of GDMSs via nice sets.

Ex. [A. Zdunik, M.U., ETDS (2004)] $\mathbb{C} \ni z \mapsto f_\lambda(z) := \lambda e^z \in \mathbb{C}$

$$(0, 1/e) \longmapsto \mathrm{HD}\big(J_r(f_\lambda)\big)$$

is real–analytic, where $J_r(f_\lambda)$ is the set of points that do not escape to infinity:

$$J_r(f_\lambda) := \Big\{z \in J(f_\lambda) : \liminf_{n\to\infty} |f_\lambda^n(z)| < +\infty\Big\}.$$

Conformal Measures

A Borel probability measure m is called t-conformal iff

$$m(f(A)) = \int_A |f'|^t \, dm$$

whenever A is Borel in the domain of f and $f|_A$ is 1–to–1.
It is a fixed of the dual operator $\mathcal{L}_t^*$:

$$\mathcal{L}^*_{-t\log|f'|} m = m.$$

Existence (and non–existence) and uniqueness (and non–uniqueness):

1. Fuchsian and Kleinian Groups [S. Patterson, 1976] and [D. Sullivan, 1980]
2. Rational functions [D. Sullivan, 1980]
3. Conformal iterated function systems and graph directed Markov systems [D. Mauldin, M.U. (1996, 2003)].
4. Transcendental meromorphic functions [K. Barański, B. Karpińska, J. Kotus, V. Mayer, A. Zdunik, M.U.]

Conformal Measures

Each non–zero finite Hausdorff and packing measure (after normalization) is conformal.

Conformal measures form a very useful tool to show that Hausdorff and/or packing measures are non–zero and finite.

Packing measure is a dual concept to Hausdorff measure introduced in late 1980s by C. Tricot, J. Taylor, and D. Sullivan. When defining Hausdorff measure one looks at minimizing covers while to define packing measure one maximizes packings:

$$\sum_i r_i^t, \quad d(x_i, x_j) > r_i + r_j \ \ (i \neq j)$$

In Euclidean spaces, packing of a set A is just a collection $\{B(x_i, r_i)\}$ of disjoint balls centered at A.

Conformal Measures

A central, perhaps the central, goal of having conformal measures, is to understand geometric measures: Hausdorff and packing. The key issue is to grasp the behavior of the ratio

$$\frac{m(B(x,r))}{r^h}$$

Ahlfors measure:

$$m(B(x,r)) \asymp r^h \text{ (and of full support)}$$

1. Then h–Hausdorff and packing measures are positive and finite, and both are equivalent to m.
2. All fractal dimensions (Hausdorff, packing, box-counting) are equal to h.
3. This is the case for expanding and subexpanding rational functions.

Conformal Measures

For parabolic rational functions the situation is different [M. Denker, M.U, ETDS (1992)]:

1. If $h = \mathrm{HD}(J(f)) > 1$, then
$$0 < \mathrm{H}_h(J(f)) < +\infty \;\text{ but }\; \mathrm{P}_h(J(f)) = +\infty$$
Ex. $z \mapsto z^2 + (1/4)$.
2. If $h = \mathrm{HD}(J(f)) < 1$, then
$$0 < \mathrm{P}_h(J(f)) < +\infty \;\text{ but }\; \mathrm{H}_h(J(f)) = 0$$
Ex. any parabolic Blaschke product whose Julia set $\neq S^1$.
3. If $h = \mathrm{HD}(J(f)) = 1$, then
$$0 < \mathrm{H}_h(J(f)), \mathrm{P}_h(J(f)) < +\infty$$
and P_h is a constant multiple of H_h.

Conformal Measures

The transcendental meromorphic case:

1. For $z \mapsto \lambda \tan z$ the same picture as for parabolic rational functions [K. Barański, Fund. Mat. (1995)]
2. $\mathbb{C} \ni z \mapsto f_\lambda(z) := \lambda e^z \in \mathbb{C}$. $\mathrm{H}_h(J_r(f_\lambda)) > 0$ and finite on each horizontal strip of finite with. Packing measure locally infinite. [A. Zdunik, M.U., Michigan Math. J. (2003)]
3. For elliptic functions [J. Kotus, M.U., CUP (2023)]
   1. For any elliptic function $\mathrm{HD}(J(f)) > 1$. [Bull. London Math. Soc. (2003)]
   2. For the parabolic case:
   $$0 < \mathrm{H}_h(J(f)) < +\infty \;\text{ but }\; \mathrm{P}_h(J(f)) = +\infty$$
   3. For the subexpanding case:
   $$0 < \mathrm{H}_h(J(f)), \mathrm{P}_h(J(f)) < +\infty$$
   and P_h is a constant multiple of H_h.

Continued Fractions

$x \in (0,1)$. Continued fraction expansion

$$x = [a_1(x), a_2(x), a_3(x), \ldots] = \cfrac{1}{a_1(x) + \cfrac{1}{a_2(x) + \cfrac{1}{a_3(x) + \ldots}\ldots}}$$

$E \subset \mathbb{N}$.

$$J_E := \left\{x \in (0,1) : a_n(x) \in E \ \ \forall n \in \mathbb{N}\right\}$$

This the limit set of the conformal iterated function system

$$[0,1] \ni x \longmapsto \frac{1}{n+x}, \ \ n \in E.$$

If E finite, then $T_G|_{J_E}$ is an expanding Cantor set. Let $J_n := J_{\{1,2,\ldots,n\}}$. Using the associated Ruelle operator D. Hensley proved in [J. Number Th. (1992), cf. World Sci. (2006] that

$$\lim_{n\to\infty} n\big(1 - \mathrm{HD}(J_n)\big) = \frac{6}{\pi^2}.$$

Continued Fractions: Bonded Type

Let

$$h_n := \mathrm{HD}(J_n).$$

We know that

$$0 < \mathrm{H}_{h_n}(J_n),\ \mathrm{P}_{h_n}(J_n) < +\infty.$$

We have proved in [A. Zdunik, M.U., J. de Th. des Nombres de Bordeaux 28 (2016)] that

$$\lim_{n\to\infty} \mathrm{H}_{h_n}(J_n) = 1 = \mathrm{H}_1((0,1)).$$

This fails for general IFSs on $[0,1]$. We are currently working with R. Tryniecki and A. Zdunik on the asymptotic of $\big(1 - \mathrm{H}_{h_n}(J_n)\big)$ and on the analogous continuity result for packing measures.

Continued Fractions: Beyond Bounded Type

Theorem (D. Mauldin, M.U., Transaction AMS (1999))

1. *If* $E = 2\mathbb{N}$, *then*
$$0 < \mathrm{P}_{h_E}(J_E) < \infty \ \text{ while } \ \mathrm{H}_{h_E}(J_E) = 0$$
2. *More generally, if* $E \subsetneq \mathbb{N}$ *has bounded gaps, then* $\mathrm{HD}(J_E) > 1/2$ *and*
$$0 < \mathrm{P}_{h_E}(J_E) < \infty \ \text{ while } \ \mathrm{H}_{h_E}(J_E) = 0$$
3. $E = \{n^p : n \in \mathbb{N}\}$, $p \geq 2$. *Then*
$$0 < \mathrm{H}_{h_E}(J_E) < \infty \ \text{ while } \ \mathrm{P}_{h_E}(J_E) = \infty$$
4. *If* $E = \{a^n : n \in \mathbb{N}\}$, *then*
$$0 < \mathrm{H}_{h_E}(J_E) < \infty \ \text{ while } \ \mathrm{P}_{h_E}(J_E) = \infty$$

Thank You!